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## A NOTE CONCERNING VEBLEN'S AXIOMS FOR GEOMETRY\*

BY

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VEBLEN has given a set of axioms for geometry.† The first eight of these are as follows:‡

AXIOM I. There exist at least two distinct points.

AXIOM II. If points  $A, B, C$  are in the order  $ABC$ , they are in the order  $CBA$ .

AXIOM III. If points  $A, B, C$  are in the order  $ABC$ , they are not in the order  $BCA$ .

AXIOM IV. If points  $A, B, C$  are in the order  $ABC$ , then  $A$  is distinct from  $C$ .

AXIOM V. If  $A$  and  $B$  are any two distinct points, there exists a point  $C$  such that  $A, B, C$  are in the order  $ABC$ .

DEF. I. The *line*  $AB$  ( $A \neq B$ ) consists of  $A$  and  $B$  and all points  $X$  in one of the possible orders  $ABX, AXB, XAB$ . The points  $X$  in the order  $AXB$  constitute the *segment*  $AB$ .  $A$  and  $B$  are the *end-points* of the segment.

AXIOM VI. If points  $C$  and  $D$  ( $C \neq D$ ) lie on the line  $AB$ , then  $A$  lies on the line  $CD$ .

AXIOM VII. If there exist three distinct points, there exist three distinct points  $A, B, C$  not in any of the orders  $ABC, BCA$ , or  $CAB$ .

AXIOM VIII. If three distinct points  $A, B$  and  $C$  do not lie on the same line, and  $D$  and  $E$  are two points in the orders  $BCD$  and  $CEA$ , then a point  $F$  exists in the order  $AFB$  and such that  $D, E, F$  lie on the same line.

Axiom II may be divided into two parts which I will call Axiom II<sub>1</sub> and Axiom II<sub>2</sub>.

AXIOM II<sub>1</sub>. If  $A, B, C$  are three distinct points in the order  $ABC$ , they are in the order  $CBA$ .

AXIOM II<sub>2</sub>. If the points  $A, B, C$  are not all distinct, then, if they are in the order  $ABC$ , they are in the order  $CBA$ .

Though Veblen's Axioms I–XII, as they stand, are mutually independent,

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\* Presented to the Society, in a somewhat different form, October 26, 1907.

† These Transactions, vol. 5 (1904), pp. 343–384.

‡ Except that I have taken the liberty of inserting the word "distinct" between the words "three" and "points" in the statement of Axiom VII. Without this slight modification (or a corresponding *understanding*) Axioms I–XII are not sufficient for Euclidean geometry.

I wish to show, in the present paper, that Axiom  $II_1$  (a *part* of Axiom II) is a consequence of Axioms III and VI and Def. I while Axiom  $II_2$  (the *remainder* of Axiom II) and Axiom IV are\* both consequences of Axioms I, III, V', VI, VII, VIII and Def. I, where Axiom V' is the following modification of Axiom V.

AXIOM V'. If  $A$  and  $B$  are any two distinct points, there exists a point  $C$ , different from  $A$  and from  $B$ , such that  $A, B, C$  are in the order  $ABC$ .

THEOREM I. *Axiom  $II_1$  is a consequence of Axioms III and VI and Def. I.*

*Proof.* By hypothesis,  $ABC$ .† Hence, by Def. I,  $C$  and  $B$  are on  $AB$ . Therefore, by Axiom VI,  $A$  is on  $CB$ . Hence, by Def. I,  $ACB, CAB$  or  $CBA$ . The second of these alternatives is ruled out by Axiom III and the hypothesis  $ABC$ . Of the two remaining alternatives, if  $ACB$  were true then, by Def. I,  $A$  and  $B$  would lie on the line  $CB$  and therefore, by Axiom VI,  $C$  would be on the line  $BA$ ; and thus, by Def. I, it would be true that either  $CBA, BCA$  or  $BAC$ . But  $BCA$  is ruled out by  $ABC$  and Axiom III, while  $BAC$  is ruled out by  $ACB$  and Axiom III. Hence it must be true that  $CBA$ .

LEMMA I. From Axiom VI and Def. I it follows that if the points  $C, D, E$  are on the line  $AB$  and  $C \neq D$ , then  $E$  is on the line  $CD$ .

*Proof.* If  $C \neq E$  then, by hypothesis and Axiom VI,  $A$  is on  $CE$  and, by Def. I,  $E$  is on  $CE$ . Hence, by Axiom VI, if  $A \neq E$ , then  $C$  is on  $EA$ . Similarly if  $A \neq E$  and  $D \neq E$  then  $D$  is on  $EA$ . Hence if  $A \neq E, C \neq E, D \neq E$ , then, by Axiom VI,  $E$  is on  $CD$ . If  $E = C$  or  $E = D$  then, by Def. I,  $E$  is on  $CD$ . If  $E = A$  then, by Axiom VI,  $E$  is on  $CD$ .

LEMMA II. From Axiom III, VI, Def. I and Axiom VII it follows that if  $A$  and  $B$  are two distinct points, then there exists a point  $C$  such that  $A, B, C$  are not on the same line.

*Proof.* If there exists no such point  $C$  then, by Lemma I, every point  $C$  is on the line  $AB$ . Hence, by another application of Lemma I, if  $D, E, F$  are any three distinct points, then  $F$  is on  $DE$  and also on  $ED$  — which is contrary to Theorem I, Axiom VII and Def. I.

THEOREM II. *From Axiom I, III, V', VI, VII, VIII and Def. I it follows that, whatever points  $A$  and  $B$  may be, the order  $ABB$  is impossible, the order  $AAB$  is impossible and the order  $ABA$  is impossible.*

*Proof.* If  $B = A$  then  $ABB$  is impossible by Axiom III.

If  $B \neq A$ , then by Lemma II there exists a point  $C$  such that  $A, B$  and  $C$  are not on the same line. By Axiom V' there exists a point  $D$ , different from  $B$ , such that  $BCD$ . By Def. I,  $C$  is on the line  $BD$ . If  $A, B$  and  $D$  were col-

\* Here Axiom  $II_2$  is to be proved by showing that its hypothesis cannot be satisfied in the presence of Axioms I, III, V', VI, VII, VIII and Def. I.

† " $ABC$ ," understood as a sentence, means that  $A, B, C$  are in the order  $ABC$ .

linear then, by Lemma I,  $A$  also would be on the line  $BD$  and then, by Axiom VI,  $B$  would be on the line  $CA$  — which would be contrary to hypothesis. Hence  $A$ ,  $B$  and  $D$  are not collinear. Suppose now that  $ABB$ . Then, by Axiom VIII, there would exist a point  $E$  such that  $DEA$  and such that  $E$ ,  $C$ ,  $B$  lie on the same line (which, by Lemma I, must be the line  $BC$ ). But, since  $A$ ,  $B$  and  $D$  are not collinear, therefore, by Lemma I,  $D$  is the only point common to the lines  $AD$  and  $BC$ . Hence  $E$  must be  $D$ , and therefore  $DDA$ . Now, by Axiom V', there exists a point  $F$ , distinct from  $D$  and from  $B$ , such that  $BDF$ . By Theorem I,  $FDB$ . Moreover, since  $F$  lies on  $BD$  and is different from  $B$  and from  $D$ , and  $A$ ,  $B$  and  $D$  are non-collinear, therefore, by Lemma I,  $A$ ,  $D$ ,  $F$  are not on the same line. Hence, by Axiom VIII, since  $FDB$  and  $DDA$ , there exists a point  $K$  such that  $AKF$  and such that  $B$ ,  $D$  and  $K$  are on the same line. It may easily be seen that  $K$  is  $F$ . Hence  $AFF$ . Now  $A$ ,  $F$  and  $B$  are non-collinear and furthermore  $AFF$  and  $FDB$ . Hence, by Axiom VIII, there exists a point  $T$  such that  $BTA$  and such that  $F$ ,  $D$  and  $T$  lie on the same line. Clearly  $T$  must be  $B$ . Hence  $BBA$ . Thus if  $ABB$ , then  $BBA$  — which is an impossible combination according to Axiom III. Hence  $ABB$  is impossible.

Suppose secondly that  $A \neq B$  and  $AAB$ . Then by hypothesis, Lemma II, Def. I and Axiom V' there exists a point  $C$  such that  $A$ ,  $B$ ,  $C$  are not on the same line and a point  $D$  different from  $A$  and from  $C$  such that  $CAD$ . By Axiom VIII there exists a point  $H$  such that  $BHC$  and such that  $D$ ,  $A$  and  $H$  are on the same line. Then clearly  $H$  is  $C$  and thus  $BCC$ . But this has been shown to be impossible. Thus  $AAB$  is impossible.

Similarly it may be shown that if  $A \neq B$  and  $ABA$  then there exists a point  $D$  such that  $DAA$ . Thus  $ABA$  is impossible.

**THEOREM III.** *Axioms II and IV are consequences of Axioms I, III, V', VI, VII, VIII and Def. I.*

Theorem III is a corollary of Theorems I and II.

In the demonstration of Theorem II advantage was taken of what might perhaps be called a sort of transitivity or cyclical arrangement among the orders  $BCD$ ,  $CEA$ ,  $AFB$  of Axiom VIII. If, in this axiom, " $AFB$ " should be replaced by " $BFA$ ," no other change being made in Axioms I–XII, then, for the set of axioms, thus slightly modified, Theorem II would no longer hold. To see this consider the independence example in which points and order are those of ordinary Euclidean geometry except that, *in addition* to the ordinary ordering, points  $A$ ,  $B$ ,  $C$  are in the order  $ABC$  whenever  $A \neq B$  and  $B = C$ .

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